

GENERATING SETS OF AFFINE GROUPS OF LOW GENUS

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ABSTRACT. We describe a new algorithm for computing braid orbits on Nielsen classes. As an application we classify all families of affine genus zero systems; that is all families of coverings of the Riemann sphere by itself such that the monodromy group is a primitive affine permutation group.

1. INTRODUCTION

Let G be a finite group. By a G -curve we mean a compact, connected Riemann surface X of genus g such that $G \leq \text{Aut}(X)$. By a G -cover we mean the natural projection π of G -curve X to its orbifold X/G . In our situation X/G is a Riemann surface of genus g_0 and π is a branched cover. We are interested in Hurwitz spaces which are moduli spaces of G -covers. By $\mathcal{H}_r^{\text{in}}(G, g_0)$ we mean the Hurwitz space of equivalence classes of G -covers which are branched over r points, such that $g(X/G) = g_0$. We are mostly interested in the case $g_0 = 0$ in which case we will simply write $\mathcal{H}_r^{\text{in}}(G)$.

Hurwitz spaces are used to study the moduli space \mathcal{M}_g of curves of genus g . For example Hurwitz himself showed the connectedness of \mathcal{M}_g by first showing that every curve admits a simple cover onto $P^1\mathbb{C}$ and then showing that the Hurwitz space of simple covers is connected.

The study of Hurwitz spaces is also closely related to the inverse problem of Galois theory. The precise connection was given by Fried and Völklein in [7].

Theorem 1 (Fried-Völklein). *The following are true:*

- (1) $\mathcal{H}_r^{\text{in}}(G)$ is an affine algebraic set which is defined over \mathbb{Q} .
- (2) If G is a group with $Z(G) = 1$, then there exists a Galois extension of $\mathbb{Q}(x)$, regular over \mathbb{Q} , with Galois group isomorphic to G and with r branch points if and only if $\mathcal{H}_r^{\text{in}}(G)$ has a \mathbb{Q} -rational point. (This also holds if \mathbb{Q} is replaced throughout by any field of characteristic 0).

The space $\mathcal{H}_r^{\text{in}}(G)$ admits an action of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Thus a \mathbb{Q} -rational point of $\mathcal{H}_r^{\text{in}}(G)$ must lie in an irreducible component which is defined over \mathbb{Q} .

Hurwitz spaces are covering spaces. In our situation, where $X/G \cong P^1\mathbb{C}$, the base space of $\mathcal{H}_r^{\text{in}}(G)$ is the configuration space of $P^1\mathbb{C}$ with r marked points. That is the space

$$\begin{aligned} D_r &:= \{S \subset P^1\mathbb{C} : |S| = r\} / \text{PGL}_2(\mathbb{C}) \\ &= (P^1\mathbb{C} \setminus \{0, 1, \infty\})^{r-3} \setminus \Delta_{r-3}. \end{aligned}$$

Where

$$\Delta_{r-3} := \{(x_1, \dots, x_{r-3}) \in (P^1\mathbb{C} \setminus \{0, 1, \infty\})^{r-3} : \exists i, j \text{ with } x_i = x_j\}.$$

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The fundamental group of D_r is the Hurwitz braid group on r -strings and is a quotient of the Artin braid group. Thus the connected components of $\mathcal{H}_r^{in}(G)$ are the orbits of the fundamental group of D_r on the fibres. We define $\mathcal{T}_r(G)$ to be those elements $(\tau_1, \dots, \tau_r) \in G^r$ such that

$$G = \langle (\tau_1, \dots, \tau_r) \rangle,$$

$$\prod_{i=1}^r \tau_i = 1,$$

and

$$|G|(\sum_{i=1}^r (|\tau_i| - 1)/|\tau_i|) = 2(|G| + g - 1).$$

The fibres of $\mathcal{H}_r^{in}(G)$ are parametrized by elements of $\underline{\tau} \in \mathcal{F}_r(G) := \mathcal{T}_r(G)/G$, where the action of G on G^r , and hence on $\mathcal{T}_r(G)$, is via diagonal conjugation. The action of the fundamental group of D_r on $\mathcal{T}_r(G)$ is the well known action of the Artin braid group which we will define in the next section. This action commutes with the action of G via diagonal conjugation and hence induces a well defined action on $\mathcal{F}_r(G)$. From the definition of the action it is clear that the action of the Artin braid group preserves the set of conjugacy classes $C_i := \tau_i^G$ of elements of $\tau_i \in \underline{\tau}$. For an r -tuple of conjugacy classes C_1, \dots, C_r we define the subset

$$\mathcal{N}i(C_1, \dots, C_r) := \{\underline{\tau} \in \mathcal{F}_r(G) : \exists \sigma \in S_r \text{ such that } \tau_i \in C_{i\sigma} \text{ for all } i\},$$

called the *Nielsen class* of C_1, \dots, C_r . The braid group action on $\mathcal{F}_r(G)$ preserves Nielsen classes which implies that connected components of $\mathcal{H}_r^{in}(G)$ are parametrized by braid orbits on Nielsen classes. The subset $\mathcal{H}(G, C_1, \dots, C_r)^{in} \subset \mathcal{H}(G)_r^{in}$ of G -curves X with $g(X/G) = 0$ is a union of components parametrized by $\mathcal{N}i(C_1, \dots, C_r)$. By slight abuse of notation it is also called a Hurwitz space.

Generally it is very difficult to determine the number of braid orbits on Nielsen classes and hence not too much is known in general. There is the celebrated result of Clebsch, alluded to above, where he shows that if $G = S_n$ and all elements of $\underline{\tau}$ are transpositions, then the corresponding Hurwitz space $\mathcal{H}(G, C_1, \dots, C_r)^{in}$ is connected. His result was recently generalized by Liu and Osserman [17] who show that if $G = S_n$ and C_i is represented by g_i where each g_i is a single cycle of length $|g_i|$, then $\mathcal{H}(G, C_1, \dots, C_r)$ is connected.

On the other hand, Fried [3] showed that if $G = A_n$, $g > 0$ and all C_i are represented by 3-cycles, then $\mathcal{H}(G, g_0, C_1, \dots, C_r)$, the space of G curves with $g_0 = g(X/G)$ and ramification in classes C_1, \dots, C_r , has one component if $g_0 = 0$ and two components if $g_0 > 0$. In the latter case the components are separated by the lifting invariant.

Finally we mention the theorem of Conway-Parker which shows that if the Schur multiplier of G is generated by commutators and the ramification involves all conjugacy classes of G sufficiently often, then the corresponding Hurwitz space is connected, hence defined over \mathbb{Q} .

Nevertheless deciding whether or not $\mathcal{H}(G, g_0, C_1, \dots, C_r)$ is connected is still an open problem, both theoretically and algorithmically. The algorithmic difficulties are due to the fact that the length of the Nielsen classes involved grows quickly. The package BRAID developed by Magaard, Shpectorov and Völklein [19] computes braid orbits algorithmically. This package is being upgraded by James, Magaard, Shpectorov [14] to generalize to the situation of orbits of the mapping class group on the fibres of the Hurwitz space $\mathcal{H}(G, g_0, C_1, \dots, C_r)$ of G -curves X with $g(X/G) = g_0$.

TABLE 1. Affine Primitive Genus Zero Systems: Number of Components

Degree	# Group isom. types	# ramification types	#comp's r=3	#comp's r=4	#comp's r=5	#comp's r=6	#comp's r=7	#comp's total
8	2	50	29	17	9	3	1	59
16	14	86	441	69	18	2	-	530
32	1	24	169	3	-	-	-	172
64	14	34	69	-	-	-	-	69
9	4	26	14	10	2	-	-	26
27	6	20	32	2	-	-	-	34
81	4	5	6	-	-	-	-	6
25	7	19	16	3	-	-	-	19
125	1	2	8	-	-	-	-	8
49	1	2	6	-	-	-	-	6
121	1	2	10	-	-	-	-	10
Totals	55	270	800	104	29	5	1	939

In this paper we introduce an algorithm which is designed to deal with long Nielsen classes. Our idea is to represent a Nielsen class as union of direct products of shorter classes, thereby enabling us to enumerate orbits of magnitude k^3 where k is an upper bound for what our standard BRAID algorithm can handle.

As an application of our algorithm we classify all braid orbits of Nielsen classes of primitive affine genus zero systems. That is to say that we find the connected components of $\mathcal{H}(G, C_1, \dots, C_r)$ of G -curves X , where G is primitive and affine with translation subgroup N and point stabilizer H , such that $g(X/H) = 0 = g(X/G)$. Recall that G is primitive if and only if H acts irreducibly on the elementary abelian subgroup N via conjugation. Equivalently this means that G acts primitively on the right cosets of N via right multiplication and N acts regularly on them. We compute that there are exactly 939 braid orbits of primitive affine genus zero systems with $G''' \neq 0$. The distribution in terms of degree and number of branch points is given in Table 1. This completes the work of Neubauer on the affine case of the Guralnick-Thompson conjecture.

Strictly speaking our new algorithm is not needed to settle the classification of braid orbits of Nielsen classes of primitive affine genus zero systems. However the problem is a good test case for our algorithm both as a debugging tool and as comparison for speed. Indeed our new algorithm shortens run times of BRAID by several orders of magnitude.

The paper is organized as follows. In Section 2 we describe our algorithm and illustrate it with an example that stresses the effectiveness. In Section 3 we discuss how we find all Hurwitz loci of affine primitive genus zero systems, displayed in tables at the end.

2. THE ALGORITHM

We begin by recalling some basic definitions. The *Artin braid group* B_r has the following presentation in terms of generators

$$\{Q_1, Q_2, \dots, Q_{r-1}\}$$

and relations

$$Q_i Q_{i+1} Q_i = Q_{i+1} Q_i Q_{i+1};$$

$$Q_i Q_j = Q_j Q_i \quad \text{for } |i - j| \geq 2.$$

The action of B_r on G^r , or *braid action* for short, is defined for all $i = 1, 2, \dots, r-1$ via:

$$Q_i : (g_1, \dots, g_i, g_{i+1}, \dots, g_r) \rightarrow (g_1, \dots, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, \dots, g_r).$$

Evidently the braid action preserves the product $\prod_{i=1}^r g_i$ and the set of conjugacy classes $\{C_1, \dots, C_r\}$ where $C_i := g_i^G$. If the classes C_i are pairwise distinct, then B_r permutes the set $\{C_1, \dots, C_r\}$ like S_r permutes the set $\{1, \dots, r\}$ where Q_i induces the permutation $(i, i+1)$. Thus we see that B_r surjects naturally onto S_r with kernel $B^{(r)}$, the *pure Artin braid group*.

We note that the group $B^{(r)}$ is generated by the elements

$$\begin{aligned} Q_{ij} &:= Q_{j-1} \cdots Q_{i+1} Q_i^2 Q_{i+1}^{-1} \cdots Q_{j-1}^{-1} \\ &= Q_i^{-1} \cdots Q_{j-2}^{-1} Q_{j-1}^2 Q_{j-2} \cdots Q_i, \end{aligned}$$

for $1 \leq i < j \leq r$.

If P is a partition of $\{1, \dots, r\}$ with stabilizer $S_P \leq S_r$, then we denote by B_P the inverse image of S_P in B_r . The group S_P is customarily called a parabolic subgroup of S_r and thus we call B_P a *parabolic subgroup* of the Artin braid group. Now B^r acts on $\mathcal{Ni}(C_1, \dots, C_r)$ permuting the classes C_i . Clearly the subgroup of B^r which preserves the order of the conjugacy classes is a parabolic subgroup of B^r and thus, from now on, we assume that the set of conjugacy classes $\{C_1, \dots, C_r\}$ is ordered in such a way that if $C_i = C_j$, then for all $i < k < j$ we have $C_k = C_i$. Let $P := P_1 \cup \dots \cup P_s$ be the partition of $\{1, \dots, r\}$ obtained by defining $x \sim y$ if $C_x = C_y$. The parabolic subgroup $B_P \leq B_r$ stabilizes the order of the classes in $\{C_1, \dots, C_r\}$.

Lemma 2. *If $\{C_1, \dots, C_r\}$ are ordered as above and P is the corresponding partition, then the set $\mathcal{Ni}^o(C_1, \dots, C_r) := \{\tau \in \mathcal{Ni}(C_1, \dots, C_r) : \tau_i \in C_i \text{ for all } i\}$ is an orbit of B_P .*

This means that the orbits of B_P on $\mathcal{Ni}^o(C_1, \dots, C_r)$ determine the components of $\mathcal{H}^{in}(G, C_1, \dots, C_r)$. As B_P -orbits are shorter than the corresponding B_r -orbits by a factor of $[S_r : S_P]$, this is a significant advantage.

For the record we note:

Lemma 3. *The set of Q_{ij} 's such that i and j lie in different blocks of P together with the Q_i 's such that i and $i+1$ lie in the same block of P is a set of generators of B_P .*

2.1. Nodes. As we noted in the introduction, Nielsen classes tend to be very large and thus we need to find ways to handle them effectively. Our algorithm achieves efficiency by interpreting tuples as elements of a Cartesian product. For this to be compatible with the action of B_P on $\mathcal{Ni}^o(C_1, \dots, C_r)$, or equivalently, with the action of $B_P \times G$ on

$$\mathcal{T}^o(C_1, \dots, C_r) := \{(\tau_1, \dots, \tau_r) \in \mathcal{T}_r(G) : \tau_i \in C_i \text{ for all } i\},$$

we need to make some additional definitions. Let $1 < k < r$ and define

$$L_k := \langle Q_i : i \leq k-1 \rangle \cap B_P,$$

$$R_k := \langle Q_i : i \geq k+1 \rangle \cap B_P.$$

Clearly $[L_k, R_k] = 1$ and every B_P -orbit on $\mathcal{Ni}^o(C_1, \dots, C_r)$ is a union of $(L_k \times R_k)$ -orbits. Equivalently every $B_P \times G$ -orbit on $\mathcal{T}^o(C_1, \dots, C_r)$ is a union of $(L_k \times R_k \times G)$ -orbits, which we call *nodes*. We refer to k as the *level*. Typically we choose k to be close to $r/2$. If (g_1, \dots, g_r) is a representative of a level k node, then we split it into its *head* (g_1, \dots, g_k) and its *tail* (g_{k+1}, \dots, g_r) . Since our package BRAID works with product 1 tuples we will identify the head and the tail with the product 1 tuples (g_1, \dots, g_k, x) and (y, g_{k+1}, \dots, g_r) respectively, where $y = \prod_{i=1}^k g_i$, $x = \prod_{i=k+1}^r g_i$. We note that $x = y^{-1}$ and that the actions of L_k and R_k centralize x and y . Hence the conjugacy class $C_x := x^G$ is an invariant of the node, which we call the *nodal type*. The following is clear.

Lemma 4. *For every node the heads of all tuples in the node form an orbit under $L_k \times G$. Similarly the tails form an $R_k \times G$ -orbit.*

We refer to the orbits above as the head (respectively, tail) orbit of the node. With the notation as above we see that the head orbit is of ramification type (C_1, \dots, C_k, C_x) and the tail orbit is of ramification type $(C_y, C_{k+1}, \dots, C_r)$. This observation allows us to determine all possible head and tail orbits independently, using BRAID. Note that subgroups generated by the head or tail may be proper in G .

Accordingly, we give the following definitions. For a ramification type $\{C_1, \dots, C_r\}$, the partition P as above, and conjugacy classes C and D of G , we define

$$\mathcal{L}_{k,C} := \{(g_1, \dots, g_k, x) : g_i \in C_i, x \in C \text{ and } 1 = (\prod_{i=1}^k g_i)x\},$$

$$\mathcal{R}_{k,D} := \{(y, g_{k+1}, \dots, g_r) : y \in D, g_i \in C_i \text{ and } 1 = y(\prod_{i=1}^k g_i)\}.$$

Note that $L_k \times G$ acts on $\mathcal{L}_{k,C}$ for all choices of C , and $R_k \times G$ acts on $\mathcal{R}_{k,D}$ for all choices of D . By slight abuse of terminology we call $(L_k \times G)$ -orbits of $\cup_C \mathcal{L}_{k,C}$ *heads* and $(R_k \times G)$ -orbits of $\cup_D \mathcal{R}_{k,D}$ *tails*. Clearly, for each node, its head orbit is among the heads and its tail orbit is among the tails. Furthermore the node is a subset of the Cartesian product of its head and its tail. We can now restate our task of finding nodes as follows. We need to find pairs of heads and tails which can correspond to nodes and then identify nodes within the Cartesian product of the head and the tail.

The first of these tasks is achieved with the following definition. A head in $\mathcal{L}_{k,C}$ *matches* a tail in $\mathcal{R}_{k,D}$ if $D = C^{-1} := \{x^{-1} : x \in C\}$. Since matching is specified entirely in terms of C and D , we note that either every head in $\mathcal{L}_{k,C}$ matches every tail in $\mathcal{R}_{k,D}$, or $\mathcal{L}_{k,C} \times \mathcal{R}_{k,D}$ contains no matching pairs. The head and tail of a node must necessarily be matching. Experiments show that most pairs of matching heads and tails lead to nodes. So no further restrictions are necessary for our algorithm.

Suppose now that $\mathcal{H} \subset \mathcal{L}_{k,C}$ and $\mathcal{T} \subset \mathcal{R}_{k,D}$ match; i.e. $D = C^{-1}$. Our task now is to find all nodes in $\mathcal{H} \times \mathcal{T}$. There are several issues that we need to address. First of all, a pair of representatives $(g_1, \dots, g_k, x) \in \mathcal{H}$ and $(y, g_{k+1}, \dots, g_r) \in \mathcal{T}$ can only give a representative of a node if $y = x^{-1}$. Therefore $\mathcal{H} \times \mathcal{T}$ is not a union of nodes; in fact most pairs of representative tuples do not work. We address this as follows.

Let us select a particular element $x_0 \in C$. A natural choice for x_0 is, for example, the minimal element of C with respect to the ordering defined in GAP [8]. Let $y_0 = x_0^{-1}$. For \mathcal{H} and \mathcal{T} as above, we define $\mathcal{H}_0 := \{(g_1, \dots, g_k, x) \in \mathcal{H} : x = x_0\}$ and $\mathcal{T}_0 := \{(y, g_{k+1}, \dots, g_r) \in \mathcal{T} : y = y_0\}$. We call \mathcal{H}_0 and \mathcal{T}_0 the *shadows* of \mathcal{H} and \mathcal{T} .

Lemma 5. *The shadows \mathcal{H}_0 and \mathcal{T}_0 are orbits for $L_k \times C_G(x_0)$ and $R_k \times C_G(x_0)$ respectively.*

Our first issue is now resolved as the representatives of \mathcal{H}_0 and \mathcal{T}_0 automatically combine to give a product 1 tuple. Furthermore for a node \mathcal{N} of type C we can similarly define the *shadow of \mathcal{N}* to be $\mathcal{N}_0 := \{(g_1, \dots, g_r) \in \mathcal{N} : \prod_{i=k+1}^r g_i = x_0\}$.

Lemma 6. *The shadow \mathcal{N}_0 is an orbit for $L_k \times R_k \times C_G(x_0)$ and furthermore it fully lies in $\mathcal{H}_0 \times \mathcal{T}_0$ where \mathcal{H}_0 and \mathcal{T}_0 are the shadows of the head and tail of \mathcal{N} .*

Thus we may work exclusively with shadows of heads, tails and nodes.

Our second issue is that combining representatives of matching head and tail shadows may not produce a tuple in $\mathcal{T}(C_1, \dots, C_r)$, because it may not generate G . We define *prenodes* as $L_k \times R_k \times G$ -orbits on

$$\{\underline{\tau} \in C_1 \times \dots \times C_r : \prod_{i=1}^r \tau_i = 1\}.$$

Clearly every node is a prenode. Our terminology, head, tail, type and shadow, extends to prenodes in the obvious way.

Lemma 7. *If \mathcal{H} and \mathcal{T} are matching heads and tails, then $\mathcal{H}_0 \times \mathcal{T}_0$ is a disjoint union of prenode shadows.*

So now our task is to identify all prenodes within $\mathcal{H}_0 \times \mathcal{T}_0$, that is to find a representative for each prenode. To achieve this, we work at the level of L_k - and R_k -orbits of \mathcal{H}_0 and \mathcal{T}_0 respectively. Let $\mathcal{O}_h \subset \mathcal{H}_0$ be an L_k -orbit and $\mathcal{O}_t \subset \mathcal{T}_0$ be an R_k -orbit. We define normalizers

$$N_{C_G(x_0)}(\mathcal{O}_h) := \{c \in C_G(x_0) : \underline{\tau}^c \in \mathcal{O}_h \text{ for all } \underline{\tau} \in \mathcal{O}_h\}$$

and

$$N_{C_G(x_0)}(\mathcal{O}_t) := \{c \in C_G(x_0) : \underline{\sigma}^c \in \mathcal{O}_t \text{ for all } \underline{\sigma} \in \mathcal{O}_t\}.$$

Because the G -action commutes with that of L_k and R_k , it suffices to check the conditions above for just a single $\underline{\tau} \in \mathcal{O}_h$ and a single $\underline{\sigma} \in \mathcal{O}_t$, respectively.

Proposition 8. *If $\mathcal{O}_h \subset \mathcal{H}_0$ is an L_k -orbit and $\mathcal{O}_t \subset \mathcal{T}_0$ is an R_k -orbit, then the prenode shadows in $\mathcal{H}_0 \times \mathcal{T}_0$ are in one-to-one correspondence with the double cosets*

$$N_{C_G(x_0)}(\mathcal{O}_h) \backslash C_G(x_0) / N_{C_G(x_0)}(\mathcal{O}_t).$$

If $\{d_1, \dots, d_s\}$ is a set of double coset representatives, then a set of representatives for the prenodes can be chosen as $\{(g_1, \dots, g_k, g_{k+1}^{d_i}, \dots, g_r^{d_i}) : 1 \leq i \leq s\}$, where (g_1, \dots, g_k, x_0) and $(y_0, g_{k+1}, \dots, g_r)$ are arbitrary representatives of \mathcal{O}_h and \mathcal{O}_t , respectively.

Proof. Let \mathcal{X} be the set of L_k -orbits of \mathcal{H}_0 and \mathcal{Y} be the set of R_k -orbits of \mathcal{T}_0 . Clearly $C_G(x_0)$ acts transitively on \mathcal{X} and on \mathcal{Y} with point stabilizers $N_{C_G(x_0)}(\mathcal{O}_h)$ and $N_{C_G(x_0)}(\mathcal{O}_t)$ respectively. Furthermore, the prenode shadows correspond to the $C_G(x_0)$ -orbits on $\mathcal{X} \times \mathcal{Y}$. The latter correspond to the double cosets as above. \square

So to construct all nodes we proceed as follows:

Algorithm: Find all level k nodes

- Input: A group G , conjugacy classes C_1, \dots, C_r and an integer $1 \leq k \leq r$.
- For each type C :
 - Set $D := C^{-1}$ and find all heads and tails by using BRAID [19].
 - From each head and tail select its shadow.
 - For each pair of head and tail shadows compute the normalizers and the double coset representatives as in Proposition 8.
 - For each prenode check whether or not its representative generates G . Store the prenodes that pass this test as nodes.
- Output all nodes. Nodes are sorted by their type, head, tail, and double coset representative.

We close this subsection with the observation that the sum of the lengths of the nodes is computable at this stage. This means that we have calculated $|\mathcal{T}(C_1, \dots, C_r)|$ by a method different from that of Staszewski and Völklein [25].

2.2. Edges. Our next step is to define a graph on our set of nodes whose connected components correspond to the braid orbits on the Nielsen class $\mathcal{Ni}(C_1, \dots, C_r)$.

Definition 9. Let $\Gamma_k(C_1, \dots, C_r)$ be the graph whose vertices are the level k nodes of $\mathcal{Ni}(C_1, \dots, C_r)$. We connect two nodes \mathcal{N}_1 and \mathcal{N}_2 by an edge if and only if there exists a tuple $\underline{\tau} \in \mathcal{N}_1$ and an element $Q \in B_P$ such that $\underline{\tau}Q \in \mathcal{N}_2$.

We remark that it is clear that the connected components of $\Gamma_k(C_1, \dots, C_r)$ are complete graphs and are in one-to-one correspondence with the braid orbits on the Nielsen class $\mathcal{Ni}(C_1, \dots, C_r)$.

Our algorithm for connecting vertices is as follows. Let S be the set of generators of B_P as in Lemma 2 minus those which are contained in $L_k \times R_k$. For each node \mathcal{N} we select a random tuple $\underline{\tau} \in \mathcal{N}$ and apply a randomly chosen generator $Q \in S$ to it. Using the head and tail of $\underline{\tau}Q$ we find the node \mathcal{N}' which contains it. If $\mathcal{N} \neq \mathcal{N}'$ we record the edge. We repeat this until we have s successes at \mathcal{N} . Note that this does not mean that we find s distinct neighbors for \mathcal{N} . If after a pre-specified number of tries t we have no successes then we conclude that \mathcal{N} is an isolated node; i.e. it is a B_P -orbit.

Once we have gone through all nodes, we obtain a subgraph Γ' of $\Gamma_k(C_1, \dots, C_r)$. We now find the connected components of Γ' and claim that these are likely to be identical to those of $\Gamma_k(C_1, \dots, C_r)$. Clearly if Γ' is connected, then so is $\Gamma_k(C_1, \dots, C_r)$. Hence in this case our conclusion is deterministic. In other cases our algorithm is Monte-Carlo.

Based on our experiments, the situation where $\Gamma_k(C_1, \dots, C_r)$ is connected is the most likely outcome. It is interesting that even for small values of s we tend to get that Γ' is connected whenever $\Gamma_k(C_1, \dots, C_r)$ is connected. Also t does not need to be large because if \mathcal{N} is not isolated then almost any choice of $\underline{\tau}$ and Q will produce an edge. This, together with the way we represent tuples as products of heads and tails, makes this part of the algorithm very fast.

Here is the formal description of the second part of the algorithm.

Algorithm: Finding the braid orbits

- Input: The k -nodes of $\mathcal{Ni}(C_1, \dots, C_r)$ arranged in terms their type, head, tail and double coset representative.
- Initialize the edge set E to the empty set.
- For each node \mathcal{N} :
 - Set counters c and d to 0.
 - Generate a random tuple $\underline{\tau}$ from \mathcal{N} by selecting random head and tail.
 - Apply a randomly chosen $Q \in S$ to $\underline{\tau}$.
 - Identify the node \mathcal{N}' containing $\underline{\tau}Q$ via its head and tail.
 - If $\mathcal{N} \neq \mathcal{N}'$, then
 - * Set c to $c+1$ and set d to $d+1$.
 - * Add the edge $(\mathcal{N}, \mathcal{N}')$ to E unless it is already known.
 - Else,
 - * Set c to $c+1$.
 - Repeat this until either $d = s$ or $d = 0$ and $c = t$.

- Determine and output the connected components of the graph Γ' whose vertices are the input nodes and whose edge set is E .

2.3. Type $\{1_G\}$ nodes. During the development of the algorithm we noticed that a significant number of nodes are of type $C = \{1_G\}$, often more than half of all nodes. This can be explained by the fact that $C_G(1_G) = G$ is largest among all classes. Furthermore, all computations for these nodes are substantially slower than for nodes of types not equal to C . The next lemma gives a criterion when such nodes can be disregarded.

Lemma 10. *Suppose \mathcal{N} is a prenode of type $\{1_G\}$ and $\underline{\tau}$ is its representative. Let H and T be the subgroups generated by the head and tail of $\underline{\tau}$, respectively. If H and T do not centralize each other, then the B_P -orbit containing \mathcal{N} contains also a prenode \mathcal{N}' of type not equal to $\{1_G\}$.*

Proof. Let $\underline{\tau} = (g_1, \dots, g_r)$. Then $H = \langle g_1, \dots, g_k \rangle$ and $T = \langle g_{k+1}, \dots, g_r \rangle$. We can also take a different set of generators for H , namely, the partial products $h_i = \prod_{j=i}^k g_j$, $i = 1, \dots, k$. Since H and T do not centralize each other, some h_s does not commute with some g_t , where $s \leq k$ and $t > k$. It is now straightforward to see that the pure braid Q_{st} takes $\underline{\tau}$ to a tuple, whose type is different from the type of $\underline{\tau}$. \square

This criterion is in fact exact. Indeed, it is clear that if H and T centralize each other then no pure braid (and more generally, no braid that preserves head and tail classes) can change the type. So the type within the B_P -orbit can change only if the same conjugacy class is present in the head and in the tail. However, a class that is present both in the head and in the tail must be central, and so the type still cannot change. Hence when H and T centralize each other then the type (whether identity or not) remains constant on the entire B_P orbit.

When the prenode \mathcal{N} is a node, we have $G = \langle H, T \rangle$, and so the condition in the lemma fails very rarely. Thus, in most cases we need not consider nodes of type $\{1_G\}$. This turns out to be a significant computational advantage.

2.4. An Example. Let $G = \text{AGL}_4(2)$, the group of affine linear transformations, acting on the 16 points of \mathbb{F}_2^4 ; the vector space of dimension 4 over the field of 2 elements. G has a unique conjugacy class of involutions whose elements have precisely 8 fixed points in their action on \mathbb{F}_2^4 (we call this class $2A$) and another whose elements have exactly 4 fixed points in their action on \mathbb{F}_2^4 (we call this second class $2B$). We consider the ramification type $\bar{C} = (2A, 2A, 2A, 2B, 2B, 2B)$. The structure constant for \bar{C} is 21, 267, 671, 040; i.e.

$$|\mathcal{T}(2A, 2A, 2A, 2B, 2B, 2B)| \leq 21, 267, 671, 040.$$

This yields that an upper bound for the size of the corresponding B_P -orbit is 65, 934. The available version of our package BRAID finds an orbit of this size within minutes. However, verifying that there is only one generating orbit takes days. This is due to the fact that BRAID spends most of its time searching for non-generating tuples in order to account for the full structure constant. Staszewski and Völklein [25] provided us with the function *NumberOfGeneratingNielsenTuples* which often helps to get around this problem. However, in this example the function runs out of memory on a 64G computer. On the other hand, after splitting \bar{C} across the middle into $(2A, 2A, 2A, C)$ and $(D, 2B, 2B, 2B)$ we compute heads and tails within minutes.

The step of constructing all nodes also takes little time. The group G has 24 non-identity conjugacy classes and hence we have 24 types of nodes.

TABLE 2. Time spent on generating heads and tails

half	total number of orbits	time spent	type with the most orbits
(2A,2A,2A,C)	155	2 mins	(2A,2A,2A,2A)
(D,2B,2B,2B)	619	10 mins	(4B,2B,2B,2B)

TABLE 3. Results from the function **AllMatchingPairs**

number of total pairs	most pairs	least pairs	number of types with no pairs
903	3A	4E	11

As shown in the table, our graph $\Gamma_k(2A, 2A, 2A, 2B, 2B, 2B)$ has 903 vertices. Drawing edges and checking that the graph Γ' is connected took less than 5 minutes. The result is that $\Gamma_k(2A, 2A, 2A, 2B, 2B, 2B)$ is connected, which means that the Hurwitz space $\mathcal{H}(\text{AGL}_4(2), 2A, 2A, 2A, 2B, 2B, 2B)$ is connected.

3. GENUS ZERO SYSTEMS AND THE GURANLICK-THOMPSON CONJECTURE

We now come to our main application. We recall some background. Suppose X is a compact, connected Riemann surface of genus g , and $\phi : X \rightarrow P^1\mathbb{C}$ is meromorphic of degree n . Let $B := \{x \in P^1\mathbb{C} : |\phi^{-1}(x)| < n\}$ be the set of branch points of ϕ . It is well known that B is a finite set and that if $b_0 \in P^1\mathbb{C} \setminus B$, then the fundamental group $\pi_1(P^1\mathbb{C} \setminus B, b_0)$ acts transitively on $F := \phi^{-1}(b_0)$ via path lifting. The image of the action of $\pi_1(P^1\mathbb{C} \setminus B, b_0)$ on F is called the *monodromy group* of (X, ϕ) and is denoted by $\text{Mon}(X, \phi)$.

We are interested in the structure of the monodromy group when the genus of X is less than or equal to two and ϕ is indecomposable in the sense that there do not exist holomorphic functions $\phi_1 : X \rightarrow Y$ and $\phi_2 : Y \rightarrow P^1\mathbb{C}$ of degree less than the degree of ϕ such that $\phi = \phi_1 \circ \phi_2$. The condition that X is connected implies that $\text{Mon}(X, \phi)$ acts transitively on F , whereas the condition that ϕ is indecomposable implies that the action of $\text{Mon}(X, \phi)$ on F is primitive.

Our first question relates to a conjecture made by Guralnick and Thompson [12] in 1990. By $cf(G)$ we denote the set of isomorphism types of the composition factors of G . In their paper [12] Guralnick and Thompson defined the set

$$\mathcal{E}^*(g) = \left(\bigcup_{(X, \phi)} cf(\text{Mon}(X, \phi)) \right) \setminus \{A_n, \mathbb{Z}/p\mathbb{Z} : n > 4, p \text{ a prime}\}$$

where $X \in \mathcal{M}(g)$, the moduli space of curves of genus g , and $\phi : X \rightarrow P^1(\mathbb{C})$ is meromorphic. They conjectured that $\mathcal{E}^*(g)$ is finite for all $g \in \mathbb{N}$. Building on work of Guralnick-Thompson [12], Neubauer [23], Liebeck-Saxl [15], and Liebeck-Shalev [16], the conjecture was established in 2001 by Frohardt and Magaard [4].

The set $\mathcal{E}^*(0)$ is distinguished in that it is contained in $\mathcal{E}^*(g)$ for all g . Moreover the proof of the Guralnick-Thompson conjecture shows that it is possible to compute $\mathcal{E}^*(0)$ explicitly.

The idea of the proof of the Guralnick-Thompson conjecture is to employ Riemann's Existence Theorem to translate the geometric problem to a problem in group theory as

follows. If $\phi : X \rightarrow P^1\mathbb{C}$ is as above with branch points $B = \{b_1, \dots, b_r\}$, then the set of elements $\alpha_i \in \pi_1(P^1\mathbb{C} \setminus B, b_0)$, each represented by a simple loop around b_i , forms a canonical set of generators of $\pi_1(P^1\mathbb{C} \setminus B, b_0)$. Let g is the genus of X . We denote by σ_i the image α_i in $\text{Mon}(X, \phi) \subset S_F \cong S_n$. Thus we have that

$$\text{Mon}(X, \phi) = \langle \sigma_1, \dots, \sigma_r \rangle \subset S_n$$

and that

$$\prod_{i=1}^r \sigma_i = 1.$$

Moreover the conjugacy class of σ_i in $\text{Mon}(X, \phi)$ is uniquely determined by ϕ . Recall that the index of a permutation $\sigma \in S_n$ is equal to the minimal number of factors needed to express σ as a product of transpositions. The Riemann-Hurwitz formula asserts that

$$2(n + g - 1) = \sum_{i=1}^r \text{ind}(\sigma_i).$$

Definition 11. If $\tau_1, \dots, \tau_r \in S_n$ generate a transitive subgroup G of S_n such that $\prod_{i=1}^r \tau_i = 1$ and $2(n + g - 1) = \sum_{i=1}^r \text{ind}(\tau_i)$ for some $g \in \mathbb{N}$, then we call (τ_1, \dots, τ_r) a genus g system and G a genus g group. We call a genus g system (τ_1, \dots, τ_r) primitive if the subgroup of S_n it generates is primitive.

If X and ϕ are as above, then we say that $(\sigma_1, \dots, \sigma_r)$ is the genus g system induced by ϕ .

Theorem 12 (Riemann Existence Theorem). For every genus g system (τ_1, \dots, τ_r) in S_n , there exists a Riemann surface Y and a cover $\phi' : Y \rightarrow P^1\mathbb{C}$ with branch point set B such that the genus g system induced by ϕ' is (τ_1, \dots, τ_r) .

Definition 13. Two covers (Y_i, ϕ_i) , $i = 1, 2$ are equivalent if there exist holomorphic maps $\xi_1 : Y_1 \rightarrow Y_2$ and $\xi_2 : Y_2 \rightarrow Y_1$ which are inverses of one another, such that $\phi_1 = \xi_1 \circ \phi_2$ and $\phi_2 = \xi_2 \circ \phi_1$.

The Artin braid group acts via automorphisms on $\pi_1(P^1\mathbb{C} \setminus B, b_0)$. We have that all sets of canonical generators of $\pi_1(P^1\mathbb{C} \setminus B, b_0)$ lie in the same braid orbit. Also the group G acts via diagonal conjugation on genus g generating sets. The diagonal and braiding actions on genus g generating sets commute and preserve equivalence of covers; that is, if two genus g generating sets lie in the same orbit under either the braid or diagonal conjugation action, then the corresponding covers given by Riemann's Existence Theorem are equivalent. We call two genus g generating systems *braid equivalent* if they are in the same orbit under the group generated by the braid action and diagonal conjugation. We have the following result, see for example [26], Proposition 10.14.

Theorem 14. Two covers are equivalent if and only if the corresponding genus g systems are braid equivalent.

Suppose now that (τ_1, \dots, τ_r) is a primitive genus g system of S_n . Express each τ_i as a product of a minimal number of transpositions; i.e. $\tau_i := \prod_j \sigma_{i,j}$. The system $(\sigma_{1,1}, \dots, \sigma_{r,s})$ is a primitive genus g system generating S_n consisting of precisely $2(n + g - 1)$ transpositions. By a famous result of Clebsch, see Lemma 10.15 in [26], any two primitive genus g systems of S_n are braid equivalent. Thus we see that every genus g system can be obtained from one of S_n which consists entirely of transpositions.

Thus, generically we expect primitive genus g systems in S_n to generate either A_n or S_n . We define $PE^*(g)_{n,r}$ to be the braid equivalence classes of genus g systems (τ_1, \dots, τ_r) in S_n such that $G := \langle \tau_1, \dots, \tau_r \rangle$ is a primitive subgroup of S_n with $A_n \cap G \neq$

A_n . We also define $G\mathcal{E}^*(g)_{n,r}$ to be the conjugacy classes of primitive subgroups of S_n which are generated by a member of $P\mathcal{E}^*(g)_{n,r}$.

We also define

$$P\mathcal{E}^*(g) := \cup_{(n,r) \in \mathbb{N}^2} P\mathcal{E}^*(g)_{n,r},$$

and similarly

$$G\mathcal{E}^*(g) := \cup_{(n,r) \in \mathbb{N}^2} G\mathcal{E}^*(g)_{n,r}.$$

We note that the composition factors of elements of $G\mathcal{E}^*(g)$ are elements of $\mathcal{E}^*(g)$.

While our ultimate goal is to determine $P\mathcal{E}^*(g)$ where $g \leq 2$, we focus here on the case $g = 0$.

Our assumption that $G = \text{Mon}(X, \phi)$ acts primitively on F is a strong one and allows us to organize our analysis along the lines of the Aschbacher-O’Nan-Scott Theorem exactly as was done in the original paper of Guralnick and Thompson [12]. We recall the statement of the Aschbacher-O’Nan-Scott Theorem from [12]

Theorem 15. *Suppose G is a finite group and H is a maximal subgroup of G such that*

$$\bigcap_{g \in G} H^g = 1.$$

Let Q be a minimal normal subgroup of G , let L be a minimal normal subgroup of Q , and let $\Delta = \{L = L_1, L_2, \dots, L_t\}$ be the set of G -conjugates of L . Then $G = HQ$ and precisely one of the following holds:

- (A) L is of prime order p .
- (B) $F^*(G) = Q \times R$ where $Q \cong R$ and $H \cap Q = 1$.
- (C1) $F^*(G) = Q$ is nonabelian, $H \cap Q = 1$.
- (C2) $F^*(G) = Q$ is nonabelian, $H \cap Q \neq 1 = L \cap H$.
- (C3) $F^*(G) = Q$ is nonabelian, $H \cap Q = H_1 \times \dots \times H_t$,
where $H_i = H \cap L_i \neq 1$, $1 \leq i \leq t$.

The members of $G\mathcal{E}^*(0)$ that arise in case (C2) were determined by Aschbacher [1]. In all such examples $Q = A_5 \times A_5$. Shih [24] showed that no elements of $G\mathcal{E}^*(0)$ arise in case (B) and Guralnick and Thompson [12] showed the same in case (C1). Guralnick and Neubauer [11] showed that the elements of $G\mathcal{E}^*(0)$ arising in case (C3) all have $t \leq 5$. This was strengthened by Guralnick [9] to $t \leq 4$ and the action of L_i on the cosets of H_i is a member of $G\mathcal{E}^*(0)$. In case (C3), where L_i is of Lie type of rank one, all elements of $G\mathcal{E}^*(0)$ and $G\mathcal{E}^*(1)$ were determined by Frohardt, Guralnick, and Magaard [5], moreover they show that $t \leq 2$. In [6] Frohardt, Guralnick, and Magaard showed that if $t = 1$, L_i is classical and L_i/H_i is a point action, then $n = [L_i : H_i] \leq 10,000$. That result together with the results of Aschbacher, Guralnick and Magaard [2] show that if $t = 1$ and L_i is classical then $[L_i : H_i] \leq 10,000$. In [13] Guralnick and Shareshian show that $G \in G\mathcal{E}^*(0)_{n,r}$ if $r \geq 9$. Moreover they show that if $G \in G\mathcal{E}^*(0)_{n,r}$ with $F^*(G)$ is alternating of degree $d < n$, then $r \geq 4$ unless $|B| = 5$ and $n = d(d-1)/2$.

So for the case where $F^*(G)$ is a direct product of nonabelian simple groups a complete picture of the elements of $G\mathcal{E}^*(0)$ is emerging.

In case (A) above, the affine case, we have that $F^*(G)$ is elementary abelian and it acts regularly on F . Case (A) was first considered by Guralnick and Thompson [12]. Their results were then strengthened by Neubauer [23]. After that, case (A) has not received much attention, which is in part due to its computational complexity.

The starting point for our investigations is Theorem 1.4 of Neubauer [23].

Theorem 16 (Neubauer). *If $F^*(G)$ is elementary abelian of order p^e and $X = P^1\mathbb{C}$, then one of the following is true:*

- (1) $G'' = 1$ and $1 \leq e \leq 2$
- (2) $p = 2$ and $2 \leq e \leq 8$,
- (3) $p = 3$ and $2 \leq e \leq 4$,
- (4) $p = 5$ and $2 \leq e \leq 3$,
- (5) $p = 7$ or 11 and $e = 2$.

The groups G with $G'' = 1$ and $1 \leq e \leq 2$ are Frobenius groups and are well understood. Thus we concentrate on the affine groups of degrees

$$\{8, 16, 32, 64, 128, 256, 9, 27, 81, 25, 125, 49, 121\}.$$

Our results are recorded in the tables below. These tables were calculated in several steps which we will now outline.

Algorithm: Enumerating Primitive Genus Zero Systems of Affine Type

- Look up the primitive affine groups G of degree p^e using the GAP function `AllPrimitiveGroups(DegreeOperation, p^e)`.
- For every group G , calculate conjugacy class representatives and permutation indices.
- Using the function `RestrictedPartitions`, calculate all possible ramification types satisfying the genus zero condition of the Riemann-Hurwitz formula.
- Let $V = \mathbb{F}_p^e = F^*(G)$. For each conjugacy class representative x calculate $\dim_V(x)$ and use Scott's Theorem to eliminate those types from the previous step which can not possibly act irreducibly on V ; i.e. can not generate a primitive group.
- Calculate the character table of G and discard those types for which the class structure constant is zero.
- For each of the remaining types of length four or more use the old version of BRAID, if possible, or else run our new algorithm. For tuples of length three determine orbits via double cosets.

A few remarks are in order. First of all, the use of Scott's theorem above is best done in conjunction with a process called translation [4]. In fact, translation was crucial in handling certain types arising in degrees 128 and 256. Secondly, using BRAID on types of length 3 is meaningless as every pure braid orbit has length one. Instead, we can compute possible generating triples using double cosets of centralizers.

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TABLE 4. The Genus Zero Systems for Affine Primitive Groups of Degree 8

group	ramification type	# of orbits	largest orbit	ramification type	# of orbits	largest orbit
$AGL(1, 8)$	(3B,3B,6B)	2	1	(3A,3B,7B)	1	1
	(3A,3B,7A)	1	1	(3A,3A,6A)	2	1
$ASL(3, 2)$	(4B,3A,7B)	2	1	(4B,3A,7A)	2	1
	(4B,3A,6A)	4	1	(4B,3A,4C)	2	1
	(4B,4B,7B)	1	1	(4B,4B,7A)	1	1
	(4B,4B,6A)	2	1	(4B,4B,4C)	4	1
	(2C,4B,7B)	1	1	(2C,4B,7A)	1	1
	(2B,7B,7B)	1	1	(2B,7A,7A)	1	1
	(2B,6A,7B)	1	1	(2B,6A,7A)	1	1
	(2B,3A,3A,3A)	1	120	(2B,4C,7B)	1	1
	(2B,4C,7A)	1	1	(2B,4B,3A,3A)	1	84
	(2B,4B,4B,3A)	1	66	(2B,4B,4B,4B)	1	36
	(2B,2C,3A,3A)	1	30	(2B,2C,4B,3A)	1	24
	(2B,2C,4B,4B)	1	24	(2B,2B,3A,7B)	1	21
	(2B,2B,3A,7A)	1	21	(2B,2B,3A,6A)	1	30
	(2B,2B,3A,4C)	1	24	(2B,2B,4B,7B)	1	14
	(2B,2B,4B,7A)	1	14	(2B,2B,4B,6A)	1	24
	(2B,2B,4B,4C)	1	24	(2B,2B,2C,7B)	1	7
	(2B,2B,2C,7A)	1	7	(2B,2B,2B,3A,3A)	1	864
	(2B,2B,2B,4B,3A)	1	648	(2B,2B,2B,4B,4B)	1	456
	(2B,2B,2B,2C,3A)	1	216	(2B,2B,2B,2C,4B)	1	192
	(2B,2B,2B,2B,7B)	1	147	(2B,2B,2B,2B,7A)	1	147
	(2B,2B,2B,2B,6A)	1	216	(2B,2B,2B,2B,4C)	1	192
	(2B,2B,2B,2B,3A)	1	6480	(2B,2B,2B,2B,2B,4B)	1	4800
	(2B,2B,2B,2B,2B,2C)	1	1680	(2B,2B,2B,2B,2B,2B)	1	48960

TABLE 5. The Genus Zero Systems for Primitive Groups of Degree 25 and 125

group	ramification type	# of orbits	largest orbit	ramification type	# of orbits	largest orbit
$5^2 : 3$	(3B,3B,3B)	8	1	(3A,3A,3A)	8	1
$5^2 : 6$	(2A,3B,6B)	4	1	(2A,3A,6B)	8	1
$5^2 : S_3$	(2A,3A,10D)	1	1	(2A,3A,10C)	1	1
	(2A,3A,10B)	1	1	(2A,3A,10A)	1	1
$5^2 : D(2 * 6)$	(2A,2B,2C,3A)	1	12			
$5^2 : D(2 * 4) : 2$	(2A,2C,2D,4C)	1	1	(2A,2C,2D,4A)	1	1
$5^2 : O + (2, 5)$	(2C,4F,8A)	1	1	(2C,4E,8B)	1	1
$5^2 : ((Q_8 : 3)^2)$	(2B,3B,12B)	1	1	(2B,3B,12A)	1	1
	(2B,3A,12D)	1	1	(2B,3A,12C)	1	1
$5^2 : ((Q_8 : 3)^4)$	(4F,3A,4E)	1	1	(4D,3A,4G)	1	1
$ASL(2, 5) : 2$	(2B,3A,20D)	1	1	(2B,3A,20C)	1	1
	(2B,3A,20B)	1	1	(2B,3A,20A)	1	1
$5^3 : 4^2 : S_3$	(2B,3A,8B)	4	1	(2B,3A,8A)	4	1

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TABLE 6. The genus zero system of $AGL(4, 2)$ Part 1

Using BRAID					
ramification type	# of orbits	largest orbit	ramification type	# of orbits	largest orbit
(2B,5A,15B)	1	1	(2B,2B,3B,7B)	1	7
(2B,5A,15A)	1	1	(2B,2B,3B,7A)	1	7
(2B,5A,14B)	1	1	(2B,2B,4B,5A)	1	80
(2B,5A,14A)	1	1	(2B,2B,4B,6C)	1	96
(2B,6C,15B)	1	1	(2B,2B,6A,5A)	1	120
(2B,6C,15A)	1	1	(2B,2B,6A,6C)	1	108
(2B,6C,14B)	1	1	(2B,2D,4D,5A)	1	90
(2B,6C,14A)	1	1	(2B,2D,4D,6C)	1	78
(2D,4F,15B)	1	1	(2B,2D,3A,5A)	1	60
(2D,4F,15A)	1	1	(2B,2D,3A,6C)	1	72
(2D,6A,15B)	3	1	(2B,2B,2B,2D,5A)	1	650
(2D,6A,15A)	3	1	(2B,2B,2B,2D,6C)	1	648
(2D,6A,14B)	2	1	(4B,4B,4D)	12	1
(2D,6A,14A)	2	1	(6A,4B,4D)	18	1
(2B,2D,2D,15B)	1	15	(6A,4B,4B)	32	1
(2B,2D,2D,15A)	1	15	(6A,6A,4F)	12	1
(2B,2D,2D,14B)	1	14	(6A,6A,4B)	52	1
(2B,2D,2D,14A)	1	14	(6A,6A,6A)	72	1
(4D,4F,5A)	6	1	(2B,2D,4B,4F)	1	96
(4D,4F,6C)	4	1	(2B,2D,4B,4B)	1	216
(4D,3B,7B)	1	1	(2B,2D,6A,4F)	1	84
(4D,3B,7A)	1	1	(2B,2D,6A,4B)	1	312
(4D,4B,5A)	6	1	(2B,2D,6A,6A)	1	414
(4D,4B,6C)	12	1	(2B,4F,4D,3B)	1	24
(4D,6A,5A)	18	1	(2B,3A,4D,3B)	1	30
(4D,6A,6C)	12	1	(2B,3A,3A,3B)	1	24
(3A,4F,5A)	2	1	(2D,2D,4D,4F)	1	88
(3A,4F,6C)	4	1	(2D,2D,4D,4B)	1	192
(3A,6A,5A)	10	1	(2D,2D,4D,6A)	1	336
(3A,6A,6C)	12	1	(2D,2D,3A,4F)	1	56
(2B,2B,4F,5A)	1	30	(2D,2D,3A,6A)	1	216
(2B,2B,4F,6C)	1	30	(2B,2B,2B,4D,3B)	1	610
(2B,2B,2B,3A,3B)	1	216	(2B,2B,2D,2D,4F)	1	576

TABLE 7. The genus zero system of $AGL(4, 2)$ Part 2

Using Matching Algorithm							
ramification type	# of nodes	# of orbits	orbit length	ramification type	# of nodes	# of orbits	orbit length
(2B,2B,2D,2D,6A)	170	1	2448	(2B,2B,2B,2B,2B,3B)	107	1	1782
(2B,2D,2D,2D,4D)	63	1	1920	(2B,2B,2D,2D,4B)	151	1	1920
(2B,2B,2B,2D,2D,2D)	903	1	15168	(2B,2D,2D,2D,3A)	56	1	1512

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TABLE 8. Genus zero systems for other primitive affine groups of degree 16

Group	ramification type	# of orbits	largest orbit	Group	ramification type	# of orbits	largest orbit
$2^4 : D(2 * 5)$	(2A,5B,4C)	1	1	$2^4 . A_6$	(2B,5B,5B)	4	1
	(2A,5B,4B)	1	1		(2B,5A,5A)	4	1
	(2A,5B,4A)	1	1		(3A,3B,5B)	2	1
	(2A,5A,4C)	1	1		(3A,3B,5A)	2	1
	(2A,5A,4B)	1	1		(2B,2B,2B,5B)	2	30
	(2A,5A,4A)	1	1		(2B,2B,2B,5A)	2	30
	(2A,2A,2A,4C)	1	12		(2B,2B,3A,3B)	1	36
	(2A,2A,2A,4B)	1	12		(2B,2B,2B,2B,2B)	2	864
	(2A,2A,2A,4A)	1	12				
$(A_4 \times A_4) : 2$	(2A,6B,6C)	1	1	$2^4 : S_5$	(2C,5A,12A)	1	1
	(2A,6A,6D)	1	1		(2C,5A,8A)	1	1
	(2A,2A,3E,3A)	1	1		(2E,6C,12A)	1	1
	(2A,2A,3D,3B)	1	1		(2E,6C,8A)	1	1
$(2^4 : 5).4$	(2A,4B,8B)	1	1		(2E,4E,12A)	1	1
	(2A,4A,8A)	1	1		(2C,2E,2E,12A)	1	6
$2^4 : S_3 \times S_3$	(2E,6B,6C)	3	1		(2C,2E,2E,8A)	1	8
	(2D,2E,2E,6C)	1	12		(2D,6C,5A)	3	1
	(2C,2E,2E,6B)	1	12		(2D,4E,5A)	3	1
	(2C,2D,2E,6A)	1	3		(2C,2E,2D,5A)	1	15
	(2C,2D,2E,2E,2E)	1	48		(2E,2E,2D,6C)	1	18
$2^4.3^2 : 4$	(2C,4D,8B)	2	1		(2E,2E,2D,4E)	1	24
	(2C,4C,8A)	2	1		(2C,2E,2E,2E,2D)	1	120
	(3A,4C,4D)	3	1				
$(S_4 \times S_4) : 2$	(2E,6B,8A)	1	1	$2^4 : A_5$	(2C,5A,5B)	3	1
	(2C,4F,12A)	1	1		(2C,6C,5B)	1	1
	(2C,6C,8A)	1	1		(2C,6C,5A)	1	1
	(2E,2C,2D,8A)	1	4		(2C,6B,5B)	1	1
	(2E,2C,2C,12A)	1	2		(2C,6B,5A)	1	1
	(2F,4F,6B)	3	1		(2C,6A,5B)	1	1
	(2E,2C,2F,6B)	1	6		(2C,6A,5A)	1	1
	(2E,2C,3A,4F)	1	6		(2C,2C,2C,5B)	1	30
	(2C,2D,2F,4F)	1	12		(2C,2C,2C,5A)	1	30
	(2C,2C,2F,6C)	1	6		(2C,2C,2C,6C)	1	18
	(2E,2E,2C,2C,3A)	1	12		(2C,2C,2C,6B)	1	18
	(2E,2C,2C,2D,2F)	1	24		(2C,2C,2C,6A)	1	18
					(2C,2C,2C,2C,2C)	1	576
$AGL(2, 4)$	(2C,4C,5A)	1	1	$2^4 . A_7$	(2B,4A,14B)	2	1
	(2C,4C,15A)	1	1		(2B,4A,14A)	2	1
	(3B,4C,6C)	4	1		(2B,7B,6A)	2	1
	(2B,2C,3B,4C)	1	20		(2B,7A,6A)	2	1
$ASL(2, 4) : 2$	(2C,5A,6A)	2	1		(2B,5A,7B)	2	1
	(2B,6A,6A)	2	1		(2B,5A,7A)	2	1
	(2B,2C,2C,5A)	1	10		(3B,3A,7B)	1	1
	(2B,2B,2C,6A)	1	12		(3B,3A,7A)	1	1
	(4A,4A,3A)	2	1		(3B,4A,6A)	6	1
	(2B,2B,2B,2C,2C)	1	80		(3B,4A,5A)	10	1
$2^4 . S_6$	(2B,2B,3B,5A)	1	10		(2B,2B,2B,7B)	2	21
	(6B,4D,3B)	2	1		(2B,2B,2B,7A)	2	21
	(6B,6B,3B)	6	1		(4A,4A,4A)	24	1
	(2B,2D,4D,3B)	1	12		(2B,2B,3B,4A)	1	192
	(2B,2D,6B,3B)	1	24				
	(2B,2B,2D,2D,3B)	1	108				

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TABLE 9. The Genus Zero Systems for Affine Primitive Groups of Degree 32

group	ramification type	# of orbits	largest orbit	ramification type	# of orbits	largest orbit
ASL(5, 2)	(2D,3B,31A)	1	1	(2D,8A,6F)	16	1
	(2D,3B,31B)	1	1	(2D,12A,6F)	16	1
	(2D,3B,31C)	1	1	(2D,6E,6F)	22	1
	(2D,3B,31E)	1	1	(2D,5A,6F)	18	1
	(2D,3B,31D)	1	1	(4A,4A,6F)	12	1
	(2D,3B,31F)	1	1	(4A,3B,8A)	12	1
	(2D,3B,30A)	1	1	(4A,3B,12A)	12	1
	(2D,3B,30B)	1	1	(4A,3B,6E)	24	1
	(2D,4J,21B)	2	1	(4A,3B,5A)	18	1
	(2D,4J,21A)	2	1	(4I,3B,4J)	18	1
	(6C,3B,4J)	12	1	(2D,2D,2D,6F)	1	720
	(2B,2D,3B,4J)	1	84	(2D,2D,4A,3B)	1	624

TABLE 10. The Genus Zero Systems for Affine Primitive Groups of Degree 64

group	ramification type	# of orbits	largest orbit	ramification type	# of orbits	largest orbit
$2^6 : 3^2 : S_3$	(2E,3F,12A)	1	1	(2E,6C,12B)	1	1
$2^6 : 7 : 6$	(2E,3B,12B)	1	1	(2E,3A,12A)	1	1
$2^6 : (3^2 : 3) : D_8$	(2G,4D,6D)	3	1	(2F,4D,6E)	3	1
$2^6 : (3^2 : 3) : SD_{16}$	(2E,4G,8D)	1	1	(2E,4G,8B)	1	1
$2^6 : (6 \times GL(3, 2))$	(2F,3C,14A)	1	1	(2F,3C,14B)	1	1
$2^6 : S_7$	(2I,4N,6K)	4	1	(2I,4D,7A)	3	1
$2^6 : (GL(2, 2) \wr S_3)$	(2L,4N,6I)	4	1			
$2^6 : (GL(3, 2) \wr 2)$	(2J, 4Q, 14H)	1	1	(2J, 4Q, 14G)	1	1
	(2I, 2J, 2J, 7B)	1	1	(2I, 2J, 2J, 7A)	1	1
$2^6 : 7^2 : S_3$	(2C,3A,14C)	1	1	(2C,3A,14D)	1	1
	(2C,3A,14E)	1	1	(2C,3A,14F)	1	1
	(2C,3A,14G)	1	1	(2C,3A,14H)	1	1
$2^6 : A_7$	(2D,4F,7A)	2	1	(2D,4F,7B)	2	1
$2^6 : GL(3, 2)$	(2G,4F,8D)	1	1	(2G,4F,8B)	1	1
	(2G,4D,6C)	2	1			
$2^6 : S_8$	(2C,6L,6K)	4	1	(2C,4O,7A)	6	1
$2^6 : GO - (6, 2)$	(4H,6C,12I)	2	1	(2C,8E,12I)	6	1
AGL(6, 2)	(2B,3B,15D)	4	1	(2B,3B,15E)	4	1

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TABLE 11. The Genus Zero Systems for Primitive Groups of Degree 9

group	ramification type	# of orbits	largest orbit	ramification type	# of orbits	largest orbit
$3^2 : 4$	(2A,4A,4A)	2	1	(2A,4B,4B)	2	1
$3^2 : D(2 \times 4)$	(2C,4A,6A)	1	1	(2A,4A,6B)	1	1
	(2A,2C,2C,6A)	1	2	(2A,2A,2C,6B)	1	2
	(2A,2C,2B,4A)	1	4	(2A,2A,2C,2C,2B)	1	8
$3^2 : (2'A_4)$	(3B,4A,3E)	1	1	(3B,6B,4A)	1	1
	(3B,6A,3D)	1	1	(3A,4A,3D)	1	1
	(3A,6B,3E)	1	1	(3A,6A,4A)	1	1
	(3B,3B,3B,2A)	1	1	(3A,3A,3A,2A)	1	1
$A\Gamma L(1, 9)$	(2A,4A,8A)	1	1	(2A,4A,8B)	1	1
$AGL(2, 3)$	(2A,3C,8B)	1	1	(2A,3C,8A)	1	1
	(2A,6A,8B)	1	1	(2A,6A,8A)	1	1
	(2A,2A,2A,8B)	1	16	(2A,2A,2A,8A)	1	16
	(2A,2A,3A,3C)	1	12	(2A,2A,3A,4A)	1	12
	(2A,2A,3A,6A)	1	12	(2A,2A,2A,2A,3A)	1	216

TABLE 12. The Genus Zero Systems for Primitive Groups of Degree 27

group	ramification type	# of orbits	largest orbit	ramification type	# of orbits	largest orbit
$3^3 . A_4$	(2A,3B,9D)	1	1	(2A,3B,9B)	1	1
	(2A,3A,9C)	1	1	(2A,3A,9A)	1	1
$3^3 (A_4 \times 2)$	(2B,3D,12B)	1	1	(2B,3D,12A)	1	1
	(2A,2B,2B,3D)	1	24			
$3^3 . S_4$	(2B,4A,9B)	1	1	(2B,4A,9A)	1	1
$3^3 (S_4 \times 2)$	(2E,4A,6G)	4	1	(2B,2E,2E,4A)	1	16
$ASL(3, 3)$	(2A,3F,13D)	2	1	(2A,3F,13C)	2	1
	(2A,3F,13B)	2	1	(2A, 3F,13A)	2	1
$AGL(3, 3)$	(2C,4A,13D)	1	1	(2C,4A,13C)	1	1
	(2C,4A,13B)	1	1	(2C,4A,13A)	1	1
	(3E,6E,4A)	8	1			

TABLE 13. The Genus Zero Systems for Primitive Groups of Degree 49

group	ramification type	# of orbits	largest orbit	ramification type	# of orbits	largest orbit
$7^2 : 4$	(2A,4B,4B)	12	1	(2A,4A,4A)	12	1
$7^2 : 3 : D(2 * 4)$	(2A,4A,6C)	3	1	(2A,4A,6B)	3	1

TABLE 14. The Genus Zero Systems for Primitive Groups of Degree 81

group	ramification type	# of orbits	largest orbit	ramification type	# of orbits	largest orbit
$3^4 : (GL(1, 3) \wr S_4)$	(6S,4C,6K)	2	1			
$3^4 : (2 \times S_5)$	(6K,4A,6M)	1	1			
$3^4 : S_5$	(6E,12A,3G)	1	1			
$AGL(4, 3)$	(2C,5A,8E)	1	1	(2C,5A,8F)	1	1

TABLE 15. The Genus Zero Systems for Primitive Groups of Degree 121

group	ramification type	# of orbits	largest orbit	ramification type	# of orbits	largest orbit
$11^2 : 3$	(3A,3A,3A)	40	1	(3B,3B,3B)	40	1
$11^2 : 4$	(2A,4A,4A)	30	1	(2A,4B,4B)	30	1
$11^2 : 6$	(2A,3B,6B)	20	1	(2A,3A,6A)	30	1
$11^2 : (Q_8 : D_6)$	(2B,3A,8A)	5	1	(2B,3A,8B)	5	1